Factoring Special Forms:

Difference of Squares
Sum and Difference of Cubes
Perfect Squares

Which of These…

$A^2 - B^2$ $A^3 - B^3$

$A^3 + B^3$ $A^2 + 2AB + B^2$

$A^2 - 2AB + B^2$

…Factor to Which of These?

$(A + B)^2$ $(A + B)(A^2 - AB + B^2)$

$(A - B)(A^2 + AB + B^2)$ $(A - B)^2$

$(A + B)(A - B)$

Using these 'forms' or 'patterns' is essential when dealing with the sums and differences of cubes (there is really no other way to get these factored). These forms are also very helpful when dealing with the perfect squares or the difference of perfect squares (would you rather use 'trial and error' or 'ac-method' on $4x^2 - 25$, or just 'notice' that this trinomial can be written as $(2x)^2 - (5)^2$ and then 'know' that it factors to $(2x + 5)(2x - 5)$?). To make this work, we must know the patterns (obviously), but we must also know how to identify 'perfect squares' and 'perfect cubes'.

The purpose of this seminar is to learn about how to identify perfect squares or cubes in a polynomial, and practice deciding whether or not one of these special forms can be used.

The parenthesis used above to identify the perfect squares in $4x^2 - 25$ are a great tool for helping you decide what perfect squares a polynomial contains, and then whether or not one of the special forms can be used.

Examples:

$$9x^2 - 16 = (3x)^2 - (4)^2$$

$$8x^3 + 27 = (2x)^3 + (3)^3$$

$$27x^3 - 64 = (3x)^3 - (4)^3$$

$$4x^2 - 12x + 9 = (2x)^2 - 2(2x)(3) + (3)^2$$

$$36x^2 + 60x + 25 = (6x)^2 + 2(6x)(5) + (5)^2$$

Before we start 'using' these forms, let's take a few minutes to practice this idea of writing a term in a perfect square, or perfect cube form.

Now we can begin thinking about how to identify one of these special forms and then using this knowledge to help make factoring (or multiplying) easier for us.

One of the most common, and useful patterns in algebra is the difference of perfect squares.

$$(A)^2 - (B)^2 = (A + B)(A - B) = (A - B)(A + B)$$

<u>MUST</u> be '$-$'
($A^2 + B^2$ <u>does not factor</u>!)

One factor with '$+$' and
the other factor with '$-$'

Notice that there is no difference between writing the factored form as either $(A + B)(A - B)$ *or as* $(A - B)(A + B)$. Besides factoring (using the above equation from left to right) and multiplication (using the above equation from right to left), this pattern is used later in algebra to do things like rationalize denominators that contain square roots and eliminate imaginary numbers from a denominator to divide complex numbers (I know, this one sounds strange, stay tuned for College Algebra).

Examples:

$$x^2 - 9 = (x)^2 - (3)^2 = (x + 3)(x - 3)$$

$$9x^2 - 16 = (3x)^2 - (4)^2 = (3x + 4)(3x - 4)$$

$$100x^2 - 81 = (10x)^2 - (9)^2 = (10x + 9)(10x - 9)$$

$$25x^2 - 64 = (5x)^2 - (8)^2 = (5x + 8)(5x - 8)$$

$$4x^2 - 121 = (2x)^2 - (11)^2 = (2x + 11)(2x - 11)$$

$$9x^6 - 16 = (3x^3)^2 - (4)^2 = (3x^3 + 4)(3x^3 - 4)$$

$$(6x - 7)(6x + 7) = (6x)^2 - (7)^2 = 36x^2 - 49$$

$$(8x - 5)(8x + 5) = (8x)^2 - (5)^2 = 64x^2 - 25$$

$$(5x - 12)(5x + 12) = (5x)^2 - (12)^2 = 25x^2 - 144$$

$$(6x^4 - 11)(6x^4 + 11) = (6x^4)^2 - (11)^2 = 36x^8 - 121$$

The patterns for the perfect squares of sums and differences are also helpful to save some time.

Always '+'

$$(A)^2 + 2(A)(B) + (B)^2 = (A + B)^2$$

$$(A)^2 - 2(A)(B) + (B)^2 = (A - B)^2$$

Same Signs

Notice that *__both__* of these patterns have '$+(B)^2$' as the last term, and '$2(A)(B)$' as the middle term, and that the sign on the middle term determines the sign between A and B in the factored forms:

$$+2(A)(B) \rightarrow (A + B)^2 \quad \textit{and} \quad -2(A)(B) \rightarrow (A - B)^2$$

Examples:

$$x^2 + 4x + 4 = (x)^2 + 2(x)(2) + (2)^2 = (x + 2)^2$$

$$x^2 - 6x + 9 = (x)^2 - 2(x)(3) + (3)^2 = (x - 3)^2$$

$$4x^2 + 12x + 9 = (2x)^2 + 2(2x)(3) + (3)^2 = (2x + 3)^2$$

$$9x^2 + 12x + 4 = (3x)^2 + 2(3x)(2) + (2)^2 = (3x + 2)^2$$

$$16x^2 - 40x + 25 = (4x)^2 - 2(4x)(5) + (5)^2 = (4x - 5)^2$$

$$64x^2 - 48x + 9 = (8x)^2 - 2(8x)(3) + (3)^2 = (8x - 3)^2$$

$$(5x + 2)^2 = (5x)^2 + 2(5x)(2) + (2)^2 = 25x^2 + 20x + 4$$

$$(6x + 5)^2 = (6x)^2 + 2(6)(5) + (5)^2 = 36x^2 + 60x + 25$$

$$(2x - 5)^2 = (2x)^2 - 2(2x)(5) + (5)^2 = 4x^2 - 20x + 25$$

$$(3x - 2)^2 = (3x)^2 - 2(3x)(2) + (2)^2 = 9x^2 - 12x + 4$$

The patterns for the sum and difference of perfect cubes are necessary to factor these types of polynomials without a lengthy discussion (~two weeks of lectures) on general factoring techniques (and even with this knowledge about general factoring techniques, the patterns save a lot of time).

Opposite Signs

$$(A)^3 + (B)^3 = (A + B)[(A)^2 - (A)(B) + (B)^2]$$

$$(A)^3 - (B)^3 = (A - B)[(A)^2 + (A)(B) + (B)^2]$$

Same Signs

Always '+'

Notice that *both* of these patterns have '$+(B)^2$' as the last term in the factored forms, and that the '$+$' or '$-$' between the $(A)^3$ and $(B)^3$ affects *two different signs* in the factored forms (one matching, and one opposite).

Examples:

$$x^3 + 1 = (x)^3 + (1)^3 = (x + 1)[(x)^2 - (x)(1) + (1)^2]$$
$$= (x + 1)(x^2 - x + 1)$$

$$x^3 + 8 = (x)^3 + (2)^3 = (x + 2)[(x)^2 - (x)(2) + (2)^2]$$
$$= (x + 2)(x^2 - 2x + 4)$$

$$27x^3 + 1 = (3x)^3 + (1)^3 = (3x + 1)[(3x)^2 - (3x)(1) + (1)^2]$$
$$= (3x + 1)(9x^2 - 3x + 1)$$

$$8x^3 + 27 = (2x)^3 + (3)^3 = (2x + 3)[(2x)^2 - (2x)(3) + (3)^2]$$
$$= (2x + 1)(4x^2 - 6x + 9)$$

$$x^3 - 1 = (x)^3 - (1)^3 = (x - 1)[(x)^2 + (x)(1) + (1)^2]$$
$$= (x - 1)(x^2 + x + 1)$$

$$64x^3 - 1 = (4x)^3 - (1)^3 = (4x - 1)[(4x)^2 + (4x)(1) + (1)^2]$$
$$= (4x - 1)(16x^2 + 4x + 1)$$

$$27x^3 - 8 = (3x)^3 - (2)^3 = (3x - 2)[(3x)^2 + (3x)(2) + (2)^2]$$
$$= (3x - 2)(9x^2 + 6x + 4)$$